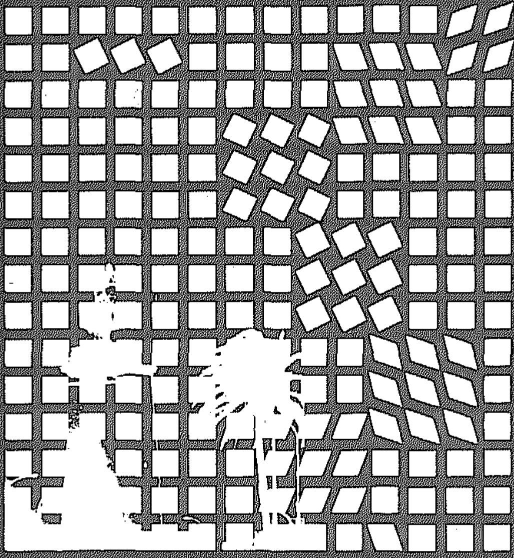
XI C.E.D.Y.A.

CONGRESO DE ECUACIONES DIFERENCIALES Y APLICACIONES

I CONGRESO DE MATEMATICA APLICADA

25 AL 29 DE SEPTIEMBRE DE 1989. FUENGIROLA (MALAGA)



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CAVITATION IN LUBRICATION: AN EVOLUTION MODEL

by

S.J.Alvarez* **J.Carrillo** and **J.I.Diaz***

Departamento de Matematica Aplicada.

Universidad Complutense. Madrid 28040.

1. Formulation.

In this communication we present some of the results of [2] on a model related to the lubrication with cavitation arising in bearings. To describe this phenomenon two unknowns are required: the pressure, p, and the relative content, γ, of the oil film. When the lubrication takes place by an incompressible fluid with convection effects, the Elrod-Adams model ([13]) leads to the following equation valid both in the cavitated and the noncavitated regions

$$(h\gamma)_t - \operatorname{div}(h^3\nabla p + h\gamma V) = 0 \quad \text{in } Q,$$

where $Q=(0,T)\times\Omega$, $\Omega\subset\mathbb{R}^2$ is a connected open set with regular boundary $\partial\Omega$, $h(t,x,y)\in C^\infty(Q)$ is a given function with $0<m\le h\le M$, and V is the given convection term (Some references on the mathematical treatment of this and others rebated models are [3] [4] [6] [7] [9] [12] and [17]). This problem can be formulated under weak regularity in the following way:

Weak Formulation. *Find* $(p,\gamma)\in L^2(0,T;H^1(\Omega))\times L^\infty(Q)$ *such that:*

i) $p\ge 0$ *and* $\gamma\in H(p)$ a.e. *in* Q *(here* H *is the Heaviside graph).*

ii) $\int_Q h\gamma\xi_t = \int_Q (h^3\nabla p\nabla\xi + h\gamma V\nabla\xi) - \int_\Omega h(0)\gamma_0\xi(0),$

$\forall\xi\in H^1(Q)$ *with* $\xi=0$ *on* $\partial_1 Q=((0,T)\times\partial\Omega)\cup(\{T\}\times\overline{\Omega})$.

iii) $p=p_e$ *on* $(0,T)\times\partial\Omega$.

2. Existence of Solution by Elliptic Regularization.

We consider the approximated problem:

Elliptic Regularized Formulation. *Find* $p^\varepsilon\in H^1(Q)$ *such that:*

i$^\varepsilon$) $p^\varepsilon\ge 0$ a.e. *in* Q .

ii$^\varepsilon$) $\varepsilon(h^3p^\varepsilon_t)_t + (hF_\varepsilon(p^\varepsilon))_t - \operatorname{div}(h^3\nabla p^\varepsilon) - \operatorname{div}(hF_\varepsilon(p^\varepsilon)V) = 0$ *in* Q

iii$^\varepsilon$) *suitable boundary conditions,*

where F_ε is a smooth function approaching the Heaviside graph.

In [2], we obtain a priori estimates for p^ε by means of a suitable election of F^ε. Then, we show the convergence:

$$p^\varepsilon \dashrightarrow p \text{ (weakly) in } L^2(0,T;H^1(\Omega))$$

$$F_\varepsilon(p^\varepsilon) \longrightarrow \gamma \text{ (weakly star) in } L^\infty(Q).$$

* Partially supported by the DGICYT project PB86/0485.

Finally, p and γ satisfy the integral equation ii) and condition i) holds in the sense $\int_Q hp(1-\gamma) = 0$. (Similar results in the literature can be found in [15],[1],[11]).

3. Uniqueness.

We shall use, now, the following approach:

Parabolic Regularized Formulation. *Find* $p^\varepsilon \in L^2(0,T;H^1(\Omega))$ *such that*:

a^ε) $p^\varepsilon \geq 0$ a.e.*in* Q .

b^ε) $(hF_\varepsilon(p^\varepsilon))_t - \operatorname{div}(h^3\nabla p^\varepsilon) - \operatorname{div}(hF_\varepsilon(p^\varepsilon)V) = 0$ *in* Q

c^ε) $p^\varepsilon = z_\varepsilon$ *on* $(0,T)\times\partial\Omega$ *and* $F_\varepsilon(p^\varepsilon(0,.)) = \gamma_0^\varepsilon(.)$ *on* Ω,

where $\gamma_0^\varepsilon > \varepsilon$, $z_\varepsilon > \varepsilon$ and F_ε is Lipschitz continuous, such that $F_\varepsilon \dashrightarrow H$ in the sense of graphs, and $F_\varepsilon(C(\varepsilon)/2) \geq 1$, (here $p_\varepsilon \geq C(\varepsilon)$ with $C(\varepsilon)$ independent on the explicit definition of F_ε).

It is not difficult to show that

$$p^\varepsilon \longrightarrow \bar{p} \text{ (weakly) in } L^2(0,T;H^1(\Omega))$$

$$F_\varepsilon(p^\varepsilon) \longrightarrow \bar{\gamma} \text{ (weakly star) in } L^\infty(Q).$$

with $(\bar{p},\bar{\gamma})$ solution of the weak formulation which we shall denote in the following as **limit solution.**

Theorem 1. *Let* $(\hat{p},\hat{\gamma})$ *be any solution of the weak formulation relative to the initial datum* $\hat{\gamma}_0$, *let* $(\bar{p},\bar{\gamma})$ *be the limit solution relative, to* $\hat{\gamma}_0$. *Then, for all* $t\geq 0$ *we have*

$$\int_\Omega h(t)\left[\hat{\gamma}(t)-\bar{\gamma}(t)\right]^+ dx \leq \int_\Omega h(0)\left[\hat{\gamma}_0-\bar{\gamma}_0\right]^+ dx$$

In particular, if $\hat{\gamma}_0 \leq \bar{\gamma}_0$ *then* $\hat{\gamma}(t) \leq \bar{\gamma}(t)$ *and so the solution of the weak formulation is unique.*

Idea of the Proof: Let $\xi \in C^\infty(Q)$, $\xi(t,.)=0$ in $\partial\Omega$ By integrating in $Q_t = (0,t)x\Omega$ for p and $\hat{p}$ and substracting we found

$$\int_\Omega h(t)\xi(t)\left[\hat{\gamma}(t)-F_\varepsilon(p^\varepsilon)\right] = \int_\Omega h(0)\xi(0)\left[\hat{\gamma}_0-\gamma_0^\varepsilon\right]^+ +$$

$$+\int_0^t\int_\Omega\left[\hat{\gamma} - F_\varepsilon(p^\varepsilon)\right]\left\{h\xi_t + \frac{\hat{p} - p_\varepsilon}{\hat{\gamma} - F_\varepsilon(p^\varepsilon)} \operatorname{div} h^3\nabla\xi - hV\nabla\xi\right\} -$$

$$-\int_0^t\int_{\partial\Omega} h^3 \frac{\partial\xi}{\partial\upsilon}(\hat{p}-p_\varepsilon).$$

Denote by $A_\varepsilon(t,x) = (\hat{p} - p_\varepsilon)(\hat{\gamma} - F_\varepsilon(p^\varepsilon))^{-1}$ and let ξ be the unique solution of the adjoint retrograde non-degenerate elliptic problem:

$$h\xi_t + A_\varepsilon(t,x)\operatorname{div}(h^3\nabla\xi) - hV\nabla\xi = 0 \quad \text{in } Q_t=(0,t)x\Omega$$

$$\xi(t)=\operatorname{sign}^{+}\left[\hat{\gamma}(t)-F_{\varepsilon}(p^{\varepsilon}(t))\right] \quad \text{on } \Omega \quad \text{(final condition)}$$
$$\xi=0 \quad \text{on} \quad (0,t)\times\partial\Omega.$$

From the maximum principle we get $0\le\xi\le1$ and using suitable barrier functions we prove:

$$\left|\int_0^t\int_{\partial\Omega} h^3\ \frac{\partial\xi}{\partial\upsilon}(\hat{p}-p_{\varepsilon})\right| = O(\varepsilon).$$

Finally

$$\int_{\Omega} h(t)\left[\hat{\gamma}(t)-\bar{\gamma}(t)\right]^{+}dx = \lim_{\varepsilon\to0}\int_{\Omega} h(t)\left[\hat{\gamma}(t)-F_{\varepsilon}(p^{\varepsilon}(t))\right]^{+}dx =$$
$$\lim_{\varepsilon\to0}\int_{\Omega} h(0)\xi(0)\left[\hat{\gamma}_0-\gamma_0^{\varepsilon}\right] \le \int_{\Omega} h(0)\left[\hat{\gamma}_0-\bar{\gamma}_0\right]^{+}. \quad \blacksquare$$

4. Time Semidiscretization.(Implicit Scheme).

We denote by α the maximal monotone graph H^{-1}. By introducing the new unknown $w=h\gamma$, one has

$$w/h \in H(p) \Leftrightarrow p\in\alpha(w/h).$$

Then, w satisfies:

$$w_t - \operatorname{div}[h^3\nabla\alpha(w/h) + wV]\ni 0 \quad \text{in } Q$$
$$w(0,x)=w_0(x) \quad \text{in } \Omega,$$

where $w_0(.)=h(0,.)\gamma_0(.)$.

Using an homogeneous discretization in time $(t_n-t_{n-1}=\lambda)$, the above equation becomes

$$\frac{w(t_n)-w(t_{n-1})}{\lambda} - \operatorname{div}\left[h^3(t_n)\nabla\alpha\left[\frac{w(t_n)}{h(t_n)}\right] + w(t_n)V(t_n)\right]\ni 0$$

From this implicit scheme we lead to the following family of problems

$$w - \lambda\operatorname{div}\left[h^3_n\nabla\alpha\left[\frac{w}{h_n}\right] + wV\right]\ni f \quad \text{in } \Omega$$
$$\alpha\left[\frac{w}{h_n}\right]\ni p_e \quad \text{on } \partial\Omega,$$

where f is a given bounded function.

Now we define the family of abstract operators

$$A_n(w) = -\operatorname{div}\left[h^3_n\nabla\alpha\left[\frac{w}{h_n}\right] + wV\right]$$

and

$$D(A_n) = \left\{ w\in L^2 : 0\le\frac{w}{h_n}\le 1,\ \alpha\left[\frac{w}{h_n}\right]\in H^1(\Omega),\ Aw\in L^2,\ \alpha\left[\frac{w}{h_n}\right]=p_e \text{ on } \partial\Omega\right\}.$$

We have

Theorem 2. *There exist a positive constant* k *depending on* V *such that if* $\lambda\le k$ *and* $f\in L^{\infty}$ *then there exist an unique* $w\in D(A_n)$ *satisfying* $w+\lambda A_n w\ni f$.

Moreover, if w_1 *and* w_2 *are solutions for* f_1 *and* f_2 *respectively then*

$$\left\| (w_1-w_2)^+ \right\|_{L^1(\Omega)} \le \left\| (f_1-f_2)^+ \right\|_{L^1(\Omega)}.$$

Idea of the Proof. We approach α for α_ε Lipschitz continuous and such that $\alpha_\varepsilon \longrightarrow \alpha$ in the sense of graphs. We get existence for the problem associated to α_ε by using pseudo-monotone operators ([16]). The above L^1 estimate is now proved, in that case, by multipliyng by $\mathrm{sign}_+(w_1-w_2)$.
The uniqueness for the regularized problems is obtained again by a duality method, generalizing a preliminary result of [18]. ▨

Results on the convergence, when λ goes to zero, are given in [2] in two following cases:

1) h and V are time-independent. We get strong convergence in $L^1(\Omega)$ by using an abstract result of [8].
2) h and V depend on t in a "regular" way. The L^1 convergence is obtained by showing a suitable condition on the resolvent operator [14]).

5. Monotonicity in time of γ

Under suitable conditions we prove the monotonicity of the free boundary defined as $\partial[p=0]$.

Theorem 3. *The following inequality holds:*

$$\mathrm{div}(Vh)(\chi_0-\gamma)-Vh\nabla\gamma-h_t(\chi_0-\gamma)+h\gamma_t \ \ge\ 0 \quad \text{in } Q$$

where $\chi_0 = \chi\,[p>0]$. *So, if* $V=0$ *then* $h_t(\chi_0-\gamma) \le h\gamma_t$ *and* $\gamma_t \ge 0$ *when* $h_t \le 0$.

Idea of the Proof. We take the test function $H_\varepsilon(p)\xi$ with $H_\varepsilon(p)=\min\{1,p/\varepsilon\}$ and $\xi \in C_0^\infty$ such that $\mathrm{supp}\,\xi(.,x)\subset(\tau_0,T-\tau_0)$, $\tau_0>0$. Then,

$$-\int_Q h_t H_\varepsilon(p)\xi = \int_Q h^3|\nabla p|^2 H'_\varepsilon \xi + \int_Q h^3\nabla p\nabla\xi H_\varepsilon(p) - \int_Q \mathrm{div}(Vh)H_\varepsilon(p)\xi$$

and hence

$$-\int_Q h_t\chi_0\xi + \int_Q \mathrm{div}(Vh)\chi_0\xi - \int_Q h^3\nabla p\nabla\xi\,\chi_0 \ \ge 0 \quad \text{for all } \xi.$$

The result follows by substracting

$$-h_t\chi_0 + \mathrm{div}(Vh)\chi_0 + \mathrm{div}(h^3\nabla p) \ \ge 0$$

and

$$-(h\gamma)_t + \mathrm{div}(Vh\gamma + h^3\nabla p) = 0. \ ▨$$

Remark. We note $H_\varepsilon(p)\xi \notin H^1(Q)$ and so the detailed proof avoids the term $[H_\varepsilon(p)]_t$. The argument is similar to the one used in [5].

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